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THE ONLY CONVEX BODY WITH EXTREMAL DISTANCE FROM THE BALL IS THE SIMPLEX

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ABSTRACT

We can extend the Banach–Mazur distance to be a distance between non-symmetric sets by allowing affine transformations instead of linear transformations. It was proved in [J] that for every convex body K we have $d(K,D) \leq n$. It is proved that if K is a convex body in $\mathbb{R}^n$ such that $d(K,D) = n$, then K is a simplex.

For every two convex bodies K_1 and K_2 in $\mathbb{R}^n$ we can define the distance between them as in [G]:

$$dK_1, K_2 \stackrel{\text{def}}{=} \inf\{\alpha |\ \exists x, y \in \mathbb{R}^n, T \in GL(n)\ \ y + K_1 \subset T(x + K_2) \subset \alpha(y + K_1)\}$$

(this is an extension of the Banach–Mazur distance for the non-symmetric case). Clearly if K_1 is an affine transformation of K_2 then $dK_1, K_2 = 1$.

It was proved in [J] that for every convex body K we have $dK, D \leq n$ and that $dK, D \leq \sqrt{n}$ for every symmetric body K . Already in [J] it was noted that these results cannot be improved because the simplex S and the unit ball of l_1 satisfy

$$dS, D = n, \qquad dB_1, D = \sqrt{n}.$$

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In [MW] it was proved that every symmetric convex body K which satisfies $dK, D = \sqrt{n}$ has a section isometric to l_1^k where k is proportional to $\log n$. In the non-symmetric case we have more rigidity: the only extremal body is the simplex. Starting with the same approach as in [MW] we get stronger conditions on K and hence a stronger result.

THEOREM 1: *If K is a convex body in $\mathbb{R}^n$ such that $dK, D = n$ then K is a simplex.*

In order to prove this theorem we need the following two lemmas.

LEMMA 1: *Let K be a convex body and set D such that D is the minimal volume ellipsoid containing K. Let $\langle\ ,\ \rangle$ be the inner product defined by D. Then for every unit vector u*

$$\left(\sup_{x\in K}\langle x,u\rangle\right)\cdot\left(-\inf_{x\in K}\langle x,u\rangle\right)\geq\frac{1}{n}.$$

Lemma 1 isn't new. It was proved by John ([J]) as a central part of the proof that $dK, D \leq n$ for every convex body K. We'll give here a simpler proof of the lemma which unfortunately doesn't give some additional information obtained in [J].

LEMMA 2: *Let K be a convex body such that $dK, D = n$. Let D be the minimal volume ellipsoid containing K and 0 the center of D. Then*

$$0\in\operatorname{conv}\left(\frac{1}{n}D\cap\partial K\right).$$

The proofs of the lemmas are technical so we will prove first the theorem and prove the lemmas later.

Proof of the theorem: Set D to be the minimal volume ellipsoid containing K and $\langle\ ,\ \rangle$ the inner product defined by D.

From Lemma 2 there are $v_1,\ldots,v_r$ and $a_1,\ldots,a_r$ such that $\|v_i\|_2=\frac{1}{n}, v_i\in\partial K$ and

$$0=\sum_{i=1}^{r}a_iv_i,\quad a_i\geq 0,\ \sum_{i=1}^{r}a_i=1.$$

By Carathéodory's theorem we can assume that $r\leq n+1$.

Set $w_i=-nv_i$; we will prove that K= $\operatorname{conv}\{w_1,\ldots,w_r\}$. For this purpose we will first prove that $w_i\in K$.

For every common point v of the boundary of a convex body K and a sphere contained in K we have

$$\forall x \in K \quad \langle x, v\rangle \leq \langle v, v\rangle.$$

So for every $i \leq r$

$$\forall x \in K \quad \langle x, v_i\rangle \leq \langle v_i, v_i\rangle = \frac{1}{n^2}.$$

Therefore

$$\forall i \leq r \ \ \forall x \in K \quad -\frac{1}{n} \ \leq \ \langle x, w_i\rangle. \tag{1}$$

We know that $\|w_i\|_2 = n \cdot \|v_i\|_2 = 1$. Using Lemma 1 by setting $u = w_i$ we get

$$\left(-\left(-\frac{1}{n}\right)\right) \sup_{x\in K}\langle w_i, x\rangle \ \geq \ \frac{1}{n}.$$

Our set K is compact so we can find $z_i \in K$ such that

$$\langle z_i, w_i\rangle = \sup_{x\in K}\langle w_i, x\rangle \geq 1.$$

The vector w_i is a unit vector and $z_i \in K \subset D$, therefore by the Cauchy – Schwartz inequality we get

$$\langle z_i, w_i\rangle \leq \|z_i\|_2 \cdot \|w_i\|_2 \leq 1 \cdot 1 = 1.$$

Thus

$$\langle z_i, w_i\rangle = 1$$

and clearly $z_i = w_i$. This means that for every i, $w_i \in K$.

We know that

$$0 = \sum_{i=1}^{r} a_i v_i$$

and therefore

$$\sum_{i=1}^{r} a_i w_i = -n \cdot 0 = 0.$$

For a fixed k we have

$$0 = \langle 0, w_k\rangle = \sum_{i=1}^{r} a_i\langle w_i, w_k\rangle = a_k\langle w_k, w_k\rangle + \sum_{i=1, i\neq k}^{r} a_i\langle w_i, w_k\rangle$$

$$= a_k + \sum_{i=1, i\neq k}^{r} a_i\langle w_i, w_k\rangle.$$

Since for every i, $w_i \in K$, we have by (1)

$$\langle w_i, w_k \rangle \geq -\frac{1}{n}. \tag{2}$$

Therefore

$$0 \geq a_k + \sum_{i=1, i\neq k}^{r} a_i(-\frac{1}{n}) = a_k + (1-a_k)\cdot(-\frac{1}{n}) = \frac{n+1}{n}a_k - \frac{1}{n}$$

and this implies

$$a_k \leq \frac{1}{n+1}. \tag{3}$$

We have $\sum_{i=1}^{r} a_i = 1, r \leq n+1$ and therefore

$$1 = \sum_{i=1}^{r} a_i \leq r\frac{1}{n+1} \leq \frac{n+1}{n+1} = 1,$$

hence

$$r = n+1,$$
$$\forall k,\ a_k = \frac{1}{n+1}.$$

This means that (3) is an equality and therefore (2) is an equality.

So we have $w_1, \dots, w_{n+1}$ such that $\|w_i\|_2 = 1$ and for every $i \neq k$, $\langle w_i, w_k \rangle = -1/n$.

Set $S = \operatorname{conv}\{w_1, \dots, w_{n+1}\}$. Then S is the simplex. K is convex and $w_i \in K$ therefore $S \subset K$. We only have to prove that $K \subset S$.

Let $x \notin S$; since $0 = \sum_{i=1}^{r} a_i w_i \in S$ there exists $0 < \lambda < 1$ such that $\lambda x \in \partial S$. Therefore λx is a convex combination of only n vectors in $\{w_1, \dots, w_{n+1}\}$. Without loss of generality we can assume that these vectors are $w_1, \dots, w_n$. Let

$$\lambda x = \sum_{i=1}^{n} b_i w_i, \qquad \sum_{i=1}^{n} b_i = 1.$$

Then

$$\langle x, w_{n+1} \rangle = \frac{1}{\lambda}\langle \lambda x, w_{n+1} \rangle = \frac{1}{\lambda}\sum_{i=1}^{n} b_i \langle w_i, w_{n+1} \rangle$$

$$= \frac{1}{\lambda}\sum_{i=1}^{n} b_i \cdot (-\frac{1}{n}) = \frac{1}{\lambda}\cdot(-\frac{1}{n}) < -\frac{1}{n}$$

and therefore by (1) $x \notin K$. So $K \subset S$ and we have

$$K = S. \qquad \blacksquare$$

We now prove our lemmas:

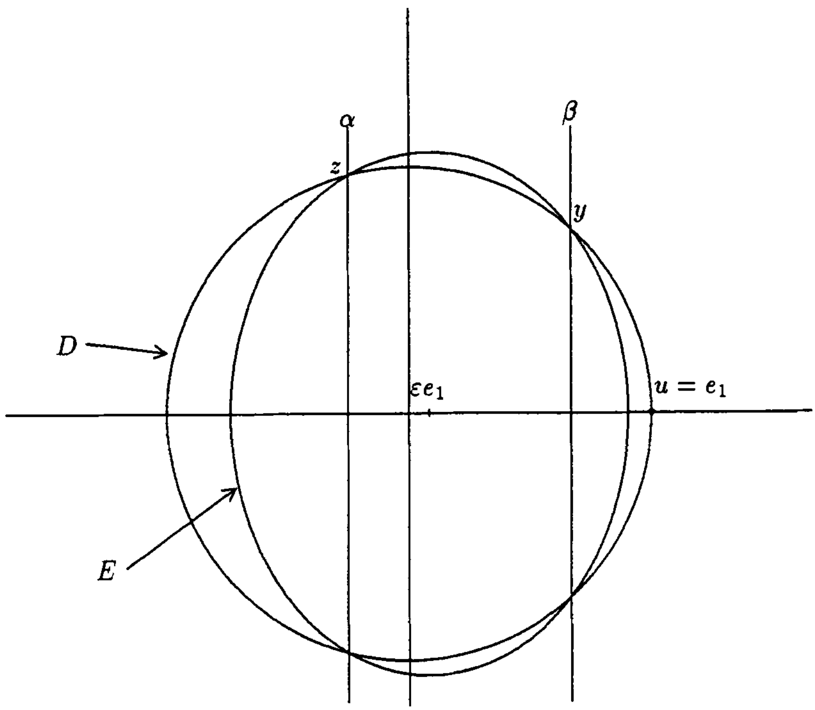
Proof of Lemma 1: Set

$$\beta = \sup_{x \in K} \langle x, u \rangle, \qquad \alpha = - \inf_{x \in K} \langle x, u \rangle.$$

We have

$$K \subset \{ x \in D \mid -\alpha \le \langle x, u \rangle \le \beta \} \stackrel{\text{def}}{=} A.$$

We will find an ellipsoid E that contains A and therefore contains K. The volume of E will have to be greater than or equal to the volume of D (since D is the minimal volume ellipsoid containing K) and this will prove that $\alpha\beta \ge \frac{1}{n}$. The drawing will help understanding how this ellipsoid is defined.



We can assume $u = e_1$; we can also assume that $\beta > \alpha > 0$ (if $\alpha > \beta$ we can use $-u$ instead of u; if $\alpha = \beta$ we can use the proof for $\beta + \delta$ instead of β for some small δ; for $\alpha < 0$ use $\alpha = \delta$ for some small δ).

Set E_ε to be an ellipsoid with center at εe_1 for small $\varepsilon > 0$:

$$E_\varepsilon = \{ \ x \mid \ a_1(\varepsilon)(x_1 - \varepsilon)^2 + a_2(\varepsilon)\sum_{i=2}^{n} x_i^2 \ \le \ 1\}$$

and set $a_1(\varepsilon), a_2(\varepsilon)$ to be such that the vectors

$$y = (\beta, \sqrt{1-\beta^2}, 0, \dots, 0),$$

$$z = (-\alpha, \sqrt{1-\alpha^2}, 0, \dots, 0)$$

are in ∂E_ε.

This means that $a_1(\varepsilon), a_2(\varepsilon)$ are determined by the following equations:

$$a_1(\varepsilon)(-\alpha - \varepsilon)^2 + a_2(\varepsilon)(1 - \alpha^2) \ = \ 1,$$

$$a_1(\varepsilon)(\beta - \varepsilon)^2 + a_2(\varepsilon)(1 - \beta^2) \ = \ 1.$$

By a simple calculation we can obtain

$$a_1(0) = a_2(0) = 1,$$

$$a_1'(0) = 2\frac{1 - \alpha\beta}{\beta - \alpha},$$

$$a_2'(0) = -2\frac{\alpha\beta}{\beta - \alpha}.$$

We'll show that $A \subset E_\varepsilon$ for small enough $\varepsilon > 0$. Indeed for every $x \in A$ we have $\|x\|_2 \le 1$ and $-\alpha \le x_1 \le \beta$. Hence

$$a_1(\varepsilon)(x_1 - \varepsilon)^2 + a_2(\varepsilon)\sum_{i=2}^{n} x_i^2 = a_1(\varepsilon)(x_1 - \varepsilon)^2 + a_2(\varepsilon)(\|x\|_2^2 - x_1^2)$$

$$\le \ a_1(\varepsilon)(x_1 - \varepsilon)^2 + a_2(\varepsilon)(1 - x_1^2).$$

Define

$$a_1(\varepsilon)(t - \varepsilon)^2 + a_2(\varepsilon)(1 - t) \stackrel{\text{def}}{=} \ell_\varepsilon(t).$$

$\ell_\varepsilon(t)$ is a polynomial of degree 2. $a_1(\varepsilon)$ and $a_2(\varepsilon)$ were chosen so that

$$\ell_\varepsilon(-\alpha) = \ell_\varepsilon(\beta) = 1.$$

Differentiating $\ell_\varepsilon(0)$ with respect to ε we get

$$\left.\frac{\partial \ell_\varepsilon(0)}{\partial \epsilon}\right|_{\varepsilon=0} = a_2'(0) < 0.$$

Hence for small $\varepsilon > 0$ we get that $\ell_\varepsilon(0) < \ell_0(0) = 1$. Thus for every $-\alpha \le t \le \beta$ we have $\ell_\varepsilon(t) \le 1$ and hence

$$a_1(\varepsilon)(x_1 - \varepsilon)^2 + a_2(\varepsilon)\sum_{i=2}^{n} x_i^2 \le \ell_\varepsilon(x_1) \le 1.$$

This proves $A \subset E_\varepsilon$.

Since $K \subset A \subset E_\varepsilon$ and D is the minimal volume ellipsoid containing K we have

$$\mathrm{vol(D)} \le \mathrm{vol(E_\varepsilon)}$$

and therefore

$$1 \ge \left(\frac{\mathrm{vol}(D)}{\mathrm{vol}(\mathrm{E}_\varepsilon)}\right)^2 = a_1(\varepsilon)\cdot a_2(\varepsilon)^{n-1} \stackrel{\text{def}}{=} v(\varepsilon).$$

Since $v(0) = 1$ we have

$$\left.\frac{\partial v(\varepsilon)}{\partial \epsilon}\right|_{\varepsilon=0} \le 0.$$

We can calculate

$$\frac{\partial v(\varepsilon)}{\partial \epsilon} = a_1'(\varepsilon)a_2(\varepsilon)^{n-1} + a_1(\varepsilon)(n-1)a_2(\varepsilon)^{n-2}a_2'(\varepsilon)$$

therefore

$$0 \ge \left.\frac{\partial v(\varepsilon)}{\partial \epsilon}\right|_{\varepsilon=0} = a_1'(0)a_2(0)^{n-1} + a_1(0)(n-1)a_2(0)^{n-2}a_2'(0)$$

$$= a_1'(0) + (n-1)a_2'(0) = 2\frac{1-\alpha\beta}{\beta-\alpha} + (n-1)\cdot\left(-2\frac{\alpha\beta}{\beta-\alpha}\right)$$

$$= \frac{2}{\beta-\alpha}(1-\alpha\beta-(n-1)\alpha\beta) = \frac{2}{\beta-\alpha}(1-n\alpha\beta).$$

Since we took $\beta > \alpha$ we have

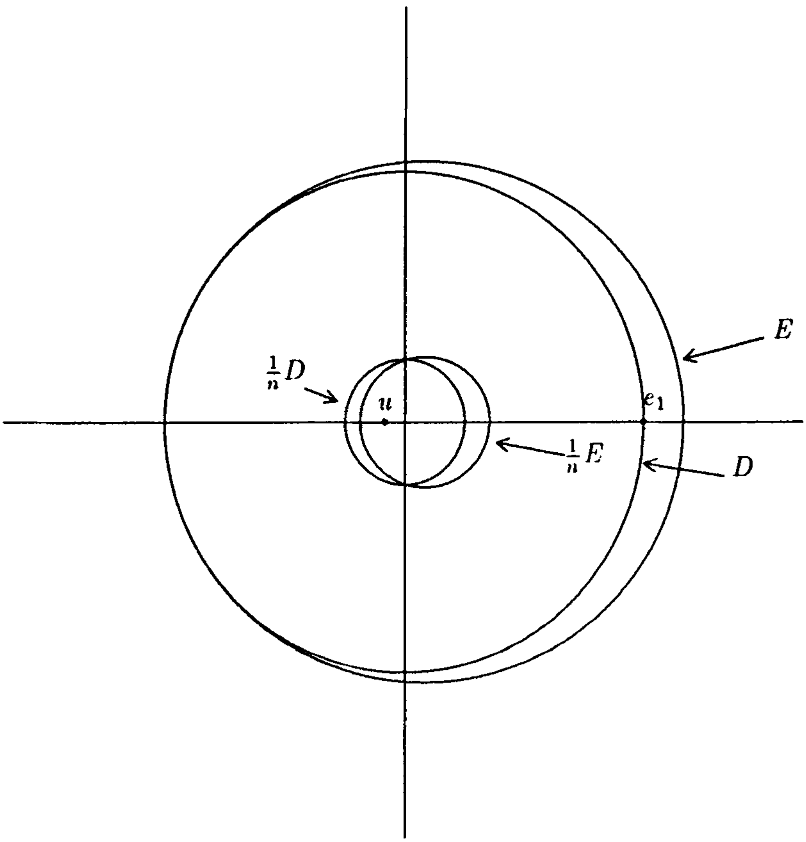
$$0 \ge 1 - n\alpha\beta,$$

$$\alpha\beta \ge \frac{1}{n}.$$

Thus

$$\left(\sup_{x\in K}\langle x,u\rangle\right)\cdot\left(-\inf_{x\in K}\langle x,u\rangle\right)\geq\frac{1}{n}. \qquad \blacksquare$$

Proof of Lemma 2: The drawing will illustrate some of the definitions in the lemma.



Let $\langle\cdot,\cdot\rangle$ be the inner product defined by D (the minimal volume ellipsoid) and $\|\cdot\|_2$ the norm defined by this inner product.

If $\partial K\cap\frac{1}{n}D=\emptyset$ then for some small $\delta>0$ we would have $(1+\delta)\frac{1}{n}D\subset K$ and therefore $dK,D\leq\frac{n}{1+\delta}$. Thus $\partial K\cap\frac{1}{n}D$ is not empty.

Suppose that $0\notin\operatorname{conv}\left(\partial K\cap\frac{1}{n}D\right)$. Let $u\in\operatorname{conv}\left(\partial K\cap\frac{1}{n}D\right)$ be the vector with the minimal norm. Set $\rho=\|u\|_2$ and set e_1 such that $u=-\rho e_1$ ($\rho>0$,

$\|e_1\|_2 = 1$). It is clear that under these definitions

$$\forall x \in \operatorname{conv}\left(\partial K \cap \frac{1}{n}D\right) \quad \langle x, u\rangle \geq \langle u, u\rangle = \rho^2. \tag{4}$$

We will show that under these conditions we can find an ellipsoid E and $\delta > 0$ such that

$$(1+\delta)\frac{1}{n}E \subset K \subset E$$

and therefore $dK, D \leq \frac{n}{1+\delta} < n$, which is a contradiction.

For every $\varepsilon > 0$ set E_ε to be an ellipsoid with center at εe_1 (ε will be determined later):

$$E_\varepsilon \stackrel{\text{def}}{=} \{x \mid \frac{1}{(1+\varepsilon)^2}(x_1-\varepsilon)^2 + \frac{1}{1+\varepsilon}\sum_{i=2}^{n} x_i^2 \leq 1\}.$$

By direct computation

$$K \subset D \subset E_\varepsilon\ . \tag{5}$$

We will show that for some small $\varepsilon > 0$ we have $\frac{1}{n}E_\varepsilon \subset \operatorname{int}(K)$. In order to prove that we need the next two sublemmas:

SUBLEMMA 2.1: *Under the definition of u and ρ we have*

$$\rho \ \geq \ \frac{1}{n^2}.$$

SUBLEMMA 2.2: *For every $\varepsilon > 0$ and for every x such that $x \in \frac{1}{n}E_\varepsilon$ and $\|x\|_2 \geq \frac{1}{n}$ we have*

$$\langle x, e_1\rangle \ \geq \ -1 + \sqrt{1 - \frac{1}{n^2}}.$$

We'll prove these sublemmas after the proof of the lemma.

Combining Sublemma 2.1 and (4) we get

$$\forall x \in \operatorname{conv}\left(\partial K \cap \frac{1}{n}D\right) \quad \langle x, e_1\rangle = -\frac{1}{\rho}\langle x, u\rangle \leq -\frac{1}{\rho}\langle u, u\rangle = -\rho \leq -\frac{1}{n^2}.$$

Since $\frac{1}{n}E_0 = \frac{1}{n}D$ and the transformation $\varepsilon \mapsto E_\varepsilon$ is continuous we get that for every $\mu > 0$ there is $\varepsilon > 0$ such that

$$\forall x \in \partial K \cap \frac{1}{n}E_\varepsilon \quad \langle x, e_1\rangle \leq -\frac{1}{n^2} + \mu.$$

We know that $\frac{1}{n}D \subset K$; therefore if $x \in \partial K$ then $\|x\|_2 \geq \frac{1}{n}$; applying Sublemma 2.2 we get that

$$\forall x \in \partial K \cap \frac{1}{n}E_\varepsilon \quad \langle x, e_1\rangle \geq -1 + \sqrt{1 - \frac{1}{n^2}}.$$

Taking $\mu > 0$ such that $-\frac{1}{n^2} + \mu < -1 + \sqrt{1 - \frac{1}{n^2}}$ we will get contradicting inequalities for every $x \in \partial K \cap \frac{1}{n}E_\varepsilon$ and thus

$$\partial K \cap \frac{1}{n}E_\varepsilon = \emptyset.$$

Since $0 \in K$ and $0 \in \frac{1}{n}E_\varepsilon$ we must have

$$\frac{1}{n}E_\varepsilon \subset \mathrm{int}(K).$$

Hence for some small $\delta > 0$

$$(1+\delta)\frac{1}{n}E_\varepsilon \subset K.$$

Combining this with (5) we get

$$(1+\delta)\frac{1}{n}E_\varepsilon \subset K \subset E_\varepsilon$$

and therefore

$$dK, D \leq \frac{n}{1+\delta} < n,$$

which contradicts the conditions of the lemma.

Therefore we must have

$$0 \in \operatorname{conv}\left(\partial K \cap \frac{1}{n}D\right). \qquad \blacksquare$$

Proof of Sublemma 2.1: We know that

$$u = \sum_{i=1}^{r} a_i v_i, \quad a_i \geq 0, \quad \sum_{i=1}^{r} a_i = 1, \quad v_i \in \partial K \cap \frac{1}{n}D. \tag{6}$$

Set $w_i = -nv_i$, we will prove that $w_i \in K$ (we will use the same arguments as in the proof of the theorem). For every common point v of the boundary of a convex body K and a sphere contained in K we have

$$\forall x \in K \quad \langle x, v\rangle \leq \langle v, v\rangle.$$

So for every $i \leq r$

$$\langle x, v_i \rangle \leq \langle v_i, v_i \rangle = \frac{1}{n^2} \tag{7}$$

and therefore

$$\langle x, w_i \rangle \geq (-n)\frac{1}{n^2} = -\frac{1}{n}.$$

Then by Lemma 1 (w_i is a unit vector)

$$\sup_{x \in K} \langle x, w_i \rangle \geq 1.$$

Therefore there exists $z_i \in K$ such that

$$\langle z_i, w_i \rangle \geq 1$$

but $z_i \in K \subset D$ and $\|w_i\|_2 = 1$ and hence $z_i = w_i$.

So for every i, we get that $w_i \in K$.

Applying (7) for some $w_j \in K$ we get that

$$\langle w_j, v_i \rangle \leq \frac{1}{n^2}$$

and hence

$$\langle v_j, v_i \rangle \geq -\frac{1}{n^3}. \tag{8}$$

Using (4) and (6) we get

$$\langle u, u \rangle = \sum_{i=1}^{r} a_i \langle v_i, u \rangle \geq \sum_{i=1}^{r} a_i \langle u, u \rangle = \langle u, u \rangle$$

and hence for every $i \leq r$

$$\langle v_i, u \rangle = \langle u, u \rangle = \rho^2.$$

Thus all the v_i's are in the same $n-1$ dimensional hyperplane. Using Carathéodory's theorem we can have $r \leq n$. Hence there exists some k such that $a_k \geq \frac{1}{n}$. Using this k in the previous equality and using (6) we have

$$\rho^2 = \langle v_k, u \rangle = \langle v_k, \sum_{i=1}^{r} a_i v_i \rangle = \sum_{i=1}^{r} a_i \langle v_k, v_i \rangle$$

$$= a_k \langle v_k, v_k \rangle + \sum_{i=1, i \neq k}^{r} a_i \langle v_k, v_i \rangle$$

using (8)

$$\geq a_k \frac{1}{n^2} + \sum_{i=1, i\neq k}^{r} a_i \cdot (-\frac{1}{n^3})$$

$$= a_k \frac{1}{n^2} - \frac{1}{n^3}(1-a_k) = a_k(\frac{1}{n^2} + \frac{1}{n^3}) - \frac{1}{n^3}$$

by our choice of k

$$\geq \frac{1}{n}(\frac{1}{n^2} + \frac{1}{n^3}) - \frac{1}{n^3} = \frac{1}{n^4}.$$

Therefore

$$\rho \geq \frac{1}{n^2}$$

and the sublemma is proved. ∎

Proof of Sublemma 2.2: Let x be a vector such that $\|x\|_2 \geq \frac{1}{n}$ and $x \in \frac{1}{n}E_\varepsilon$. Set $x_i = \langle x, e_i \rangle$. Then

$$\sum_{i=1}^{n} x_i^2 \ \geq \ \frac{1}{n^2},$$

$$\frac{1}{(1+\varepsilon)^2}(x_1 - \varepsilon)^2 + \frac{1}{1+\varepsilon}\sum_{i=2}^{n} x_i^2 \ \leq \ \frac{1}{n^2}.$$

Combining the last two inequalities we get

$$\frac{1}{(1+\varepsilon)^2}(x_1 - \varepsilon)^2 + \frac{1}{1+\varepsilon}(\frac{1}{n^2} - x_1^2) \ \leq \ \frac{1}{n^2}.$$

And by simple calculations

$$0 \ \leq \ x_1^2 + 2x_1 + \frac{1}{n^2} + \frac{\varepsilon}{n^2} - \varepsilon.$$

Since $\frac{\varepsilon}{n^2} - \varepsilon \leq 0$ we have

$$0 \ \leq \ x_1^2 + 2x_1 + \frac{1}{n^2}.$$

The roots of the $x_1^2 + 2x_1 + \frac{1}{n^2}$ are $-1 \pm \sqrt{1 - \frac{1}{n^2}}$ so we have

$$x_1 \leq -1 - \sqrt{1 - \frac{1}{n^2}} \quad or \quad x_1 \geq -1 + \sqrt{1 - \frac{1}{n^2}}.$$

Since $x \in \frac{1}{n}E_\varepsilon$ we have $x_1 \geq -\frac{1}{n}$ so clearly

$$x_1 \geq -1 + \sqrt{1 - \frac{1}{n^2}}$$

and the sublemma is proved. ∎

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References

[G] B. Grünbaum, *Measures of symmetry for convex sets*, Proceedings of Symposia in Pure Mathematics. Convexity **VII** (1963), 233–270.

[J] F. John, *Extremum problems with inequalities as subsidiary conditions*, Courant Anniversary Volume, New York, 1948, pp. 187–204.

[MW] V.D. Milman and H. Wolfson, *Minkowski spaces with extremal distance from the Euclidean space*, Isr. J. Math. **29** (1978), 113–131.